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SECOND ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS SOLVER VIA MULTISTEP METHOD WITH NEWTON-CƠTES QUADRATURE

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Abstract: *This study is concerned with the technique of implementing members of Newton-Cotes quadrature as the integrator of the integral term in a second-order integro-differential equation. Despite the efficient computational cost of linear multistep solvers, the limitation of most solution techniques involving multistep methods and integration quadrature is the solution schemes' inability to admit various forms of quadrature other than the trapezoidal quadrature. This study examines the relationship between choices of the multistep method employed, a systemic partitioning of the independent parameter domain, and the admissible quadrature for various instances. Thus, for this experiment, we considered an efficient 4-step block Multistep Method and an auxiliary 3-step block Method, both derived from an orthogonal polynomial as a basis function. To determine the suitability of our composition as numerical solvers, the multistep methods and corresponding quadrature are employed in solving some test problems. The performance of our method is determined via a comparison between computed results, theoretical results, and results from related sources. The outcome of our implementation strategy shows that our method is efficient and convergent. In addition, implementation results revealed that there is an optimal number of partitions for achieving a computation convergence. Most importantly, the study addresses the limitations encountered by most multistep methods in the literature.*

1. Introduction

This paper discusses the implementation strategy of linear multistep method with integration quadrature for solving linear second-order Fredholm integro-differential equation (LFIDE). We consider an equation of the form

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$$y''(x) = \Gamma(x, y, y') + \lambda \int_a^b k(x,t)y(t)dt \quad (1)$$
$$y(a) = y_0, y'(a) = y'_0, a \le x \le b$$

where the integral limits a and b define either Volterra or Fredholm integro-differential equations depending on the parameter b. If the upper limit of the integral is a constant, then Eq. 1 is a Fredholm type, otherwise, it is a Volterra type equation. Also, $\Gamma(x, y, y') = p(x)y'(x) + q(x)y(x) + g(x)$ defines the type of integro-differential equation, while the Kernel $k(x,t)$ may be separable and λ is a real parameter. Eq. 1 finds application in scientific and physical engineering; thus, authors have made advances in proposing numerical solution to Eq. 1.

Following the development, some methods for obtaining approximate solutions to equations in the form of Eq. 1 includes the quadrature-difference method [1], the method employed half-sweep quadrature-difference schemes with an iterative implementation technique. Similarly, [2] explored the Spline collocation method for solving Fredholm and Volterra integro-differential equations, while [3] proposed the Gauss elimination method; combining composite Simpson's $1/3$ rule and second order finite difference method as a second-order LFIDEs solver.

Kamoh et al. [4] developed a block multistep method on shifted Legendre basis function with Trapezoidal quadrature as a simultaneous solver of second-order Volterra integro-differential equations. Also, [5] proposed an approach based on the Discrete Adomian Decomposition Method for equations featuring Volterra and Fredholm integrals, incorporating Trapezoidal and Simpson's rules as numerical integrators. Recently, [6] discussed the efficiency of the Haar Wavelet Method, an approach that explored nice properties of Haar Functions, as a numerical integrator for second-order integro-differential equations. Some other notable methods for solving Eq. 1 comprise the Generalized Minimal Residual Method (GMRES) [10], CAS Wavelet method [11], compact finite method [12], Legendre collocation matrix method [13], and the differential transformation method [14].

For the purpose of our study, we considered our previously derived block multistep method for solving second-order ordinary differential equations. The method's derivation process follows the interpolation and collocation of a basis function technique as presented in [8]. Given that the solver was found to be efficient, the present setting shall implement the derived block method with members of Open and Closed Newton-Cotes integration quadrature as a solver for Fredholm equations in the form of Eq. 1. The study equally presents the derivation of a 3-step method as an auxiliary method. Since the study focuses on addressing certain difficulty for the multistep solution technique, we explored shallow properties of the derived auxiliary method, stating that it possesses sufficient properties of a convergent solver. The auxiliary method is considered to be a suitable solver for a particular case of Newton-Cotes quadrature and all integer multiples of that quadrature.

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2. Methodology

2.1 Derivation of an Efficient 4-Step Block Method

The study assumes that the solution to Eq. 1 is well approximated by a function from the family of orthogonal polynomials, such as shifted Legendre, shifted Chebyshev, or Lucas polynomials. The approximating function may be expressed as

$$y(x) = \sum_{i=0}^{m} a_i \, \phi_i(x) \quad (2)$$

Conventionally, Eq. 2 goes through interpolation and collocation techniques, to systematically determine the co-efficient $a_i's$ of Eq. 2. Thus, we considered interpolation points x_n and x_{n+2}; while points x_n, x_{n+1}, x_{n+2}, x_{n+3} and x_{n+4} were chosen as collocation points (see [8]).

Evaluating Eq. 2 and its second derivative at stated points resulted in a system of 7 equations with co-efficient a_i, $i = 0, \ldots, 6$ to be determined. The result of the determined co-efficient $a_i's$ of Eq. 2 differed with respect to the choice of polynomial $\phi_i(x)$. However, upon substituting the values of $a_0, \ldots, a_7$ in the approximating function, Eq. 2, we obtained a uniform continuous formulation for any choice of orthogonal polynomial considered. The continuous form for the method is expressed as

$$y(x) = \sum_{j=0}^{1} \alpha_{\frac{4j}{2}}(x,h) y_{n+\frac{4j}{2}} + \sum_{j=0}^{4} \beta_j(x,h) f_{n+j}\,, j = 0, 1, \ldots, 4 \quad (3)$$

where $\alpha_{\frac{4j}{2}}(x,h)$ and $\beta_j(x,h)$ are continuous in x and are given as

$$\alpha_0 = \frac{x_{n+2} - x}{2h}$$

$$\alpha_2 = \frac{x - x_n}{2h}$$

$$\beta_0 = \frac{1}{1440h^4}\left(2(x_n - x)^6 + 30h(x_n - x)^5 + 175h^2(x_n - x)^4 + 500h^3(x_n - x)^3 + 720h^4(x_n - x)^2 + 424h^5(x_n - x)\right.$$

$$\beta_1 = \frac{1}{1440h^4}\left((x_n - x)^6 + \frac{27}{2}h(x_n - x)^5 + 65h^2(x_n - x)^4 + 120h^3(x_n - x)^3 - 144h^5(x_n - x)\right)$$

$$\beta_2 = \frac{1}{120h^4}\left((x_n - x)^6 + 12h(x_n - x)^5 + \frac{95}{2}h^2(x_n - x)^4 + 60h^3(x_n - x)^3 - 20h^5(x_n - x)\right)$$

$$\beta_3 = \frac{1}{180h^4}\left((x_n - x)^6 + \frac{21}{2}h(x_n - x)^5 + 35h^2(x_n - x)^4 + 40h^3(x_n - x)^3 - 16h^5(x_n - x)\right)$$

$$\beta_4 = \frac{1}{720h^4}\left((x_n - x)^6 + 9h(x_n - x)^5 + \frac{55}{2}h^2(x_n - x)^4 + 30h^3(x_n - x)^3 - 12h^5(x_n - x)\right)$$

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To generate the desired discrete schemes, we evaluate Eq. 3 at $x = x_{n+4}$, and $x = x_{n+3}$. In addition, we differentiated Eq. 3 with respect to x, as

$$y'(x) = \sum_{j=0}^{1} \alpha'_{\frac{4j}{2}}(x,h)\, y_{n+\frac{4j}{2}} + \sum_{j=0}^{4} \beta'_j(x,h)\, f_{n+j} \quad (4)$$

and evaluate Eq. 4 at points $x = x_{n+i}$, $i = 0, \dots, 4$. From these computations, we obtained an efficient 4-Step Block Method (*4SBM*) presented as:

$$\left.\begin{aligned}
y_{n+4} - 2y_{n+2} + y_n &= \frac{h^2}{15}(f_n + 16f_{n+1} + 26f_{n+2} + 16f_{n+3} + f_{n+4}) \\
2y_{n+3} - 3y_{n+2} + y_n &= \frac{h^2}{480}(17f_n + 252f_{n+1} + 402f_{n+2} + 53f_{n+3} - 3f_{n+4}) \\
y_{n+2} - 2y_{n+1} + y_n &= \frac{h^2}{480}(19f_n + 204f_{n+1} + 14f_{n+2} + 4f_{n+3} - f_{n+4}) \\
y_{n+2} - y_n &= 2hy'_n + \frac{h^2}{90}(53f_n + 144f_{n+1} - 30f_{n+2} + 16f_{n+3} - 3f_{n+4}) \\
y_{n+2} - y_n &= 2hy'_{n+1} - \frac{h^2}{360}(39f_n + 70f_{n+1} - 144f_{n+2} + 42f_{n+3} - 7f_{n+4}) \\
y_{n+2} - y_n &= 2hy'_{n+2} - \frac{h^2}{90}(5f_n + 104f_{n+1} + 78f_{n+2} - 8f_{n+3} + f_{n+4}) \\
y_{n+2} - y_n &= 2hy'_{n+3} - \frac{h^2}{360}(31f_n + 342f_{n+1} + 768f_{n+2} + 314f_{n+3} - 15f_{n+4}) \\
y_{n+2} - y_n &= 2hy'_{n+4} - \frac{h^2}{90}(3f_n + 112f_{n+1} + 126f_{n+2} + 240f_{n+3} + 59f_{n+4})
\end{aligned}\right\} \quad (5)$$

2.2 Analysis of the Efficient *4SBM*

The study presents the necessary and sufficient requirements expected of linear multistep-type solvers of differential equations, since our interest in this setting is to demonstrate the efficiency of Equation 5 in solving a class of integro-differential equations. Salawu et al. [8] presented details of the properties of Eq. 5 and further demonstrated the accuracy of Equation 5 as an ordinary differential equation solver. Hence, we extend the superiority of the our *4SBM* solver to become an integro-differential equations solver by implementing the block of schemes in conjunction with members of Newton-Cotes quadrature.

2.2.1 Order and Error Constant of the *4SBM*

Conventionally, the linear difference operator

$$L[y(x);h] = \sum(\alpha_j y(x_n + jh) - h^2\beta_j f(x_n + jh) \quad (6)$$

is employed to determine the possible order of the method in the sense of $O(h^{p+1})$ as well as the local truncation error associated with the

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discrete schemes of the derived linear multistep method.

In light of the techniques associated with the use of Eq. 6, we found that the members of the derived *4SBM* were of order $[6,5,5,5,5,5,5,5]^T$ with respective error constants $\left[-\frac{2}{945},\frac{1}{480},\frac{4}{315},\frac{1}{480},-\frac{61}{14},\frac{2}{7},-\frac{61}{14},\frac{4}{315}\right]^T$.

2.2.2 Consistency, Zero Stability and Convergence

Since the derived schemes of the *4SBM* are of order greater than one, they are considered consistent. In addition, the roots of the first characteristic polynomial for each of the candidates comprising the *4SBM* fall within the unit circle and in each case, the spurious roots were also simple. Thus, each scheme of the block was determined to be zero stable and hence, the derived method is considered convergent (see [8]).

2.3 Integration Quadrature

The integral in Eq. 1 is approximated following the techniques of integration formulae. We thus, recall a family of quadrature in what follows.

2.3.1 Newton-Cộtes Quadrature

Given that the value of a function $f(x)$ defined on $[a,b]$ is known at $n+1$ equally spaced points $a \le x\_\{0\} \le \ldots \le x\_\{n\} \le b$, Newton-Cộtes family of integration quadrature is classified as Open or Closed, depending on whether the endpoints of the interval $[a,b]$ are incorporated in the solution process or not.

Closed Newton-Cộtes (CNC) quadrature takes $a = x_0$ and $x_n = b$, so that

$$x_i = a + ih, h = \frac{b-a}{n}$$

whereas, Open Newton- Cộtes (ONC) quadrature considers $x_0 > a$ and $x_n < b$, with

$$x_i = a + (i+1)h, h = \frac{b-a}{n+2}$$

Furthermore, in either case of CNC or ONC quadrature, the number of partitions - n is considered as an integer multiple and it determines the member of Newton-Cộtes quadrature to be implemented.

Hence, for CNC, with $n = 1,2,3,\ldots$, Trapezoidal is implementable; $n = 2,4,6,\ldots$, Simpson's $1/3$ is considered, $n = 3,6,9,\ldots$, Simpson's $3/8$ is equally considered and when $n = 4,8,12,\ldots$, Boole's rule is employed.

Similarly, for ONC, when n is a multiple of 2, Milne's Rule is defined, with $h = \frac{b-a}{n+2}$. Whereas, when $n = 3$ with $h = \frac{b-a}{5}$, an integral is approximated with the formula expressed as

$$\int_a^b f(x)dx = \frac{5h}{24}(11f_1 + f_2 + f_3 + 11f_4). \quad (7)$$

2.3 An Auxiliary 3-Step Block Method

2.3.1 Derivation of the Method

To demonstrate the comparative implementation of Simpson's $3/8$ and Milne's Rules with fewer function evaluations, we construct a 3-Step Block Method (*3SBM*). For the derivation process, we equally assumed that Eq. 2 sufficiently approximates a solution to linear differential equations in the

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form of Eq. 1. In this case, the polynomial $\phi(x_i)$ of Eq. 2 is taken to be of Lucas polynomial, and the basis function for the derivation process takes the form

$$y(x) = \sum_{i=0}^{m} a_i \, \phi_i(x) \quad (8)$$

We interpolated and collocated Eq. 8, at points $x = x_n, x_{n+1}$ and points $x = x_n, x_{n+1}, x_{n+2}$ respectively. The resulting system of 5 equations in 5 unknowns is presented as follows:

$$y_n = 2a_0 + a_1 x_n + a_2(x_n^2 + 2) + a_3(x_n^3 + 3x_n) + a_4(x_n^4 + 4x_n^2 + 2) + a_5(x_n^5 + 5x_n^3 + 5x_n)$$

$$y_{n+1} = 2a_0 + a_1 x_{n+1} + a_2(x_{n+1}^2 + 2) + a_3(x_{n+1}^3 + 3x_{n+1}) + a_4(x_{n+1}^4 + 4x_{n+1}^2 + 2) + a_5(x_{n+1}^5 + 5x_{n+1}^3 + 5x_{n+1})$$

$$f_n = 2a_2 + 6a_3 x_n + a_4(12x_n^2 + 8) + a_5(20x_n^3 + 30x_n)$$

$$f_{n+1} = 2a_2 + 6a_3 x_{n+1} + a_4(12x_{n+1}^2 + 8) + a_5(20x_{n+1}^3 + 30x_{n+1})$$

$$f_{n+2} = 2a_2 + 6a_3 x_{n+2} + a_4(12x_{n+2}^2 + 8) + a_5(20x_{n+2}^3 + 30x_{n+2})$$

$$f_{n+3} = 2a_2 + 6a_3 x_{n+3} + a_4(12x_{n+3}^2 + 8) + a_5(20x_{n+5}^3 + 30x_{n+5})$$

The system of linear equations is solved to determine the co-efficient of expansion $a_0, \dots, a_5$ using Maple. The determined values of $a_i's$ are substituted in Eq. 8, to conventionally derive the continuous form of the desired method given by

$$y(x) = \sum_{j=0}^{1} \alpha_j(x,h) y_{n+j} + \sum_{j=0}^{3} \beta_j(x,h) f_{n+j} \quad (9)$$

Given that six schemes are required to construct the block of second-order differential equations solver, we obtained the first derivative of Eq. 9 with respect to x as

$$y'(x) = \sum_{j=0}^{1} \alpha_j'(x,h) y_{n+j} + \sum_{j=0}^{3} \beta_j'(x,h) f_{n+j} \quad (10)$$

The discrete schemes of our derived 3-step block method (*3SBM*) are obtained when the continuous formulation as in Eq. 9 is evaluated at points $x = x_{n+3}$ and $x = x_{n+2}$, while Eq. 10 is evaluated at points $x = x_{n+k}, k = 0, 1, 2, 3$. This block is presented as

$$\begin{aligned}
&y_{n+3} - 3y_{n+1} + 2y_n = \frac{h^2}{12}(f_{n+3} + 12f_{n+2} + 21f_{n+1} + f_n) \\
&y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{12}(f_{n+2} + 10f_{n+1} + f_n) \\
&360y_{n+1} - 360y_n = 360hy_n + h^2(8f_{n+3} - 39f_{n+2} + 114f_{n+1} + 97f_n) \\
&360y_{n+1} - 360y_n = 360hy_{n+1} + h^2(-7f_{n+3} + 36f_{n+2} - 171f_{n+1} - 38f_n) \\
&360y_{n+1} - 360y_n = 360hy_{n+2} + h^2(8f_{n+3} - 159f_{n+2} - 366f_{n+1} - 23f_n) \\
&360y_{n+1} - 360y_n = 360hy_{n+3} + h^2(-127f_{n+3} - 444f_{n+2} - 291f_{n+1} - 38f_n)
\end{aligned} \quad (11)$$

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2.3.2 Analysis of the 3SBM

Following the same procedure discussed in Sections 2.2.1 and 2.2.2, each scheme in Eq. 11 was determined to be of order 4, that is, the method is a block of uniform order schemes. In addition, with each scheme being consistent and zero stable, we conclude that the block comprises of convergent discrete schemes, where the method is expected to minimally introduce local error $O(h^5)$.

3 Experimental Test Problems

4SBM in conjunction with various members of CNC as well as ONC quadrature shall be implemented in solving integro-differential equations of second-order. To determine performance, accuracy and ease of implementation, we make a comparison between obtained results, exact results and results from other methods. In addition, we employ the auxiliary *3SBM* to demonstrate the technique of using an appropriate n –step method when a certain number of integration partition n is consider, and also for specific members of CNC and ONC quadrature.

The solution procedure begins with generating sets of equations from Eq. 5 by evaluating Eq. 5 at $n = 0, 4, \dots, m-4$, where m is the desired number of partitions. It is expected that m should be a multiple of the step number of a derived method. Thus, the functions $f_0, f_1, \dots, f_m$ are evaluated with respect to the quadrature of interest.

Suppose we consider trapezoidal quadrature within an interval $[0, 1]$ with $m = 4$ partitions, then we evaluate Eq. 5 at $n = 0$ to give

$$\begin{aligned}
y_4 - 2y_2 + y_0 &= \frac{h^2}{15}(f_0 + 16f_1 + 26f_2 + 16f_3 + f_4)\\
2y_3 - 3y_2 + y_0 &= \frac{h^2}{480}(17f_0 + 252f_1 + 402f_2 + 53f_3 - 3f_4)\\
y_2 - 2y_1 + y_0 &= \frac{h^2}{480}(19f_0 + 204f_{n+1} + 14f_{n+2} + 4f_{n+3} - f_{n+4})\\
y_2 - y_0 &= 2hy_0' + \frac{h^2}{90}(53f_0 + 144f_{\square} - 30f_2 + 16f_3 - 3f_4)\\
y_2 - y_0 &= 2hy_1' - \frac{h^2}{360}(39f_0 + 70f_1 - 144f_2 + 42f_3 - 7f_4)\\
y_2 - y_0 &= 2hy_2' - \frac{h^2}{90}(5f_0 + 104f_1 + 78f_2 - 8f_3 + f_4)\\
y_2 - y_0 &= 2hy_3' - \frac{h^2}{360}(31f_0 + 342f_1 + 768f_2 + 314f_3 - 15f_4)\\
y_2 - y_0 &= 2hy_4' - \frac{h^2}{90}(3f_0 + 112f_1 + 126f_2 + 240f_3 + 59f_4)
\end{aligned} \tag{12}$$

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where y_0 is known. Each of the functions $f_n, n = 0, 1, 2, 3, 4$ represent the right-hand side of Eq. 1, and they are computed using

$$f_n = \Gamma(x_n, y_n, y_n') + \frac{h}{2}\left(k(x_n, t_0)y(t_0) + 2k(x_n, t_1)y(t_1) + 2k(x_n, t_2)y(t_2) + 2k(x_n, t_3)y(t_3) + k(x_n, t_4)y(t_4)\right) \quad (13)$$

In Eq. (13), the integral term of the right-hand side of Eq. 1 is approximated by the Trapezoidal quadrature, so that each f_n of Eq. 12 becomes expression in terms of y_1, y_2, y_3 and y_4. Hence, the generated set of equations as in Eq. 12 in terms y_1, y_2, y_3 and y_4, are solved simultaneously, to obtain all values $y_1, \dots y_4, y_1', \dots, y_4'$.

It is to be noted that the step size h is a function of m, that is $h = \frac{b-a}{m}$ in order to ensure that the expected number of unknowns y_n and y_n' are returned from the computation of f_n.

The following test problems were solved;

Problem 1

Consider the second-order LFIDE

$$y''(x) = e^x - \frac{4}{3}x + \int_0^1 xty(t)dt \; y(0) = 1, y'(0) = 2$$

with exact solution $y(x) = e^x + x$.

Comparative performances of the candidates of CNC quadrature are determined with our *4SBM*. In addition, we made a comparison with the results in [6]

Problem 2

Given the Fredholm IDE

$$y''(x) = g(x, y, y') + \sin(x) \int_{-1}^1 e^{-t} y(t)dt \; y(-1) = e^{-1}, y(1) = e$$

$$g(x, y, y') = xy'(x) + xy(x) + e^x - 2\sin(x)$$

The exact solution to this problem subject to the initial conditions is $y(x) = e^x$.

Similarly, we checked for the consistency of our *4SBM* and compared the rate of convergence of our method with the convergence of the method employed in [7].

Problem 3

A second-order LFIDE as obtained in [10] is

$$y''(x) = \frac{9}{4} - \frac{1}{3}x + \int_0^1 (x - t)y(t)dt, y(0) = y'(0) = 0,$$

with exact solution $y(x) = x^2$.

The efficiency of *4SBM* and the auxiliary 3SBM featuring members of CNC and ONC quadrature was demonstrated in solving this problem.

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4 Results

All test results are presented with a varied step size h, dependent upon the number of partitions, n, chosen within the solution interval. *MAPLE* calculations were performed with a precision of 10 to 20 decimal places, while tabular presentations are limited to a maximum of 10 decimal places for convenience. In what follows, the study presents the results of Problems 1 through 3.

Table 1: Result for Problem 1 with n = 32, with comparative absolute error at even grid points.

x	y (Exact)	Trapezoidal	Simpson's 1/3	Boole's
0.06250	1.1269944589	2.20488×10^{-08}	9.75531×10^{-13}	7.79086×10^{-13}
0.12500	1.2581484530	1.76395×10^{-08}	1.24850×10^{-11}	1.55191×10^{-12}
0.18750	1.3937302494	5.95337×10^{-07}	4.49689×10^{-11}	2.40579×10^{-12}
0.25000	1.5340254166	1.41117×10^{-07}	1.09051×10^{-10}	3.24471×10^{-12}
0.31250	1.6793379411	2.75620×10^{-07}	2.15159×10^{-10}	4.16818×10^{-12}
0.37500	1.8299914146	4.76271×10^{-07}	3.73929×10^{-10}	5.06795×10^{-12}
0.43750	1.9863302986	7.56301×10^{-07}	5.95776×10^{-10}	6.05344×10^{-12}
0.50000	2.1487212707	1.12893×10^{-05}	8.91350×10^{-10}	7.01426×10^{-12}
0.56250	2.3175546569	1.60741×10^{-05}	1.27105×10^{-09}	8.06768×10^{-12}
0.62500	2.4932459574	2.20495×10^{-05}	1.74554×10^{-09}	9.07980×10^{-12}
0.68750	2.6762374695	2.93480×10^{-05}	2.32520×10^{-09}	1.01973×10^{-11}
0.75000	2.8670000166	3.81017×10^{-05}	3.02071×10^{-09}	1.12647×10^{-11}
0.81250	3.0660347872	4.84429×10^{-05}	3.84245×10^{-09}	1.24487×10^{-11}
0.87500	3.2738752939	6.05041×10^{-05}	4.80110×10^{-09}	1.35736×10^{-11}
0.93750	3.4910894580	7.44174×10^{-05}	5.90700×10^{-09}	1.48289×10^{-11}
1.00000	3.7182818284	9.03151×10^{-05}	7.17090×10^{-09}	1.30161×10^{-11}

Table 2: Comparison of Maximum Absolute Error in Derived Methods with Method of [6] for Problem 1.

Method's MAE	$n = 16$	$n = 24$	$n = 32$	$n = 36$
MAE Simp. **1/3**	1.1373×10^{-07}			4.4795×10^{-09}
MAE Simp. **3/8**		5.1001×10^{-08}		
MAE Boole's			1.3016×10^{-11}	
MAE of [6] (Cp)	7.0755×10^{-05}		1.0442×10^{-05}	
MAE of [6] (Gp)	4.1703×10^{-05}		1.7757×10^{-05}	

MAE: Maximum Absolute Error Cp: collocation points Gp: Gaussian points

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Table 3: Deviation measure at selected grid points for Problem 2 with $n = 32$

x	y exact	Error in Trap.	Error in Simp. 1/3	Error in Boole's
-0.8125	0.4437473102	8.31562×10^{-11}	8.31644×10^{-11}	8.31659×10^{-11}
-0.6250	0.5352614287	1.62077×10^{-10}	1.62089×10^{-10}	1.62091×10^{-10}
-0.4375	0.6456485267	2.27527×10^{-10}	2.27539×10^{-10}	2.27541×10^{-10}
-0.2500	0.7788007833	2.64111×10^{-10}	2.64120×10^{-10}	2.64121×10^{-10}
-0.0625	0.9394130631	2.70668×10^{-10}	2.70672×10^{-10}	2.70672×10^{-10}
0.1250	1.1331484533	2.69946×10^{-10}	2.69944×10^{-10}	2.69943×10^{-10}
0.3125	1.3668379414	2.54921×10^{-10}	2.54913×10^{-10}	2.54912×10^{-10}
0.5000	1.6487212709	2.04043×10^{-10}	2.04032×10^{-10}	2.04031×10^{-10}
0.6875	1.9887374697	1.19031×10^{-10}	1.19022×10^{-10}	1.19020×10^{-10}
0.8750	2.3988752940	4.97811×10^{-11}	4.97759×10^{-11}	4.97750×10^{-11}

Table 4: Max. AE of our 4SBM with different quadrature compared with that of [7] for Problem 2

Method	Max. AE
4SBM with Trapezoidal	2.84302×10^{-10}
4SBM with Simpson **1/3**	2.84301×10^{-10}
4SBM with Boole's	2.84301×10^{-10}
ADM of [7]	2.66410×10^{-02}

Table 5: Solution to Problem 3 with varied n

n	Quadrature	Max. Abs. Error
36	Simpson 1/3	2.4816×10^{-15}
	Simpson 3/8	2.4812×10^{-15}
	Boole's	2.4816×10^{-15}
24	Simpson 1/3	2.4810×10^{-15}
	Simpson 3/8	2.4811×10^{-15}
	Boole's	2.4813×10^{-15}
12	Simpson 1/3	1.2406×10^{-15}
	Simpson 3/8	1.2406×10^{-15}
	Boole's	1.2406×10^{-15}

Table 6: Solution to Problem 3 using *4SBM* implemented with ONC for $n = 3$, $h = 0.2$

x	y exact	y computed	Absolute Error
0.2	0.04	0.0399999998	2.0×10^{-11}
0.4	0.16	0.1599999999	1.0×10^{-10}
0.6	0.36	0.3599999999	1.0×10^{-10}
0.8	0.64	0.6399999998	2.0×10^{-10}

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Table 7: Solution to Problem 3 via Trial *3SBM* with Milne's Rule for $n = 2$

x	y exact	y computed	Absolute Error
0.25	0.0625	0. 0625	0.00×10^{-10}
0.50	0.2500	0. 2500	0.00×10^{-10}
0.75	0.5625	0. 5625	0.00×10^{-10}
1.00	-	-	-

Table 8: Approximate solution to Problem 3 via Trial *3SBM* with Simpson's 3/8 Rule for $n = 3$

x	y exact	y computed	Absolute Error
0.0000	0.0000	-	-
0.3333	0.1111	0. 1111	0.00×10^{-10}
0.6666	0.4444	0. 4444	0.00×10^{-10}
0.9999	1.0000	0.9999	2.00×10^{-10}

5. Discussion

From Table 1, we observe that Boole's formula outperformed other integration formulas. However, our derived 4SBM consistently performed well with any choice of Closed Newton-Cotes integration quadrature. Table 2 reveals that our method aligns with the convergence principle of multistep methods. This is demonstrated by Simpson's 1/3, which mildly exhibited improved performance with more partitions. Convergence of our method (with fewer partitions) is evident when compared with the method of Rohul et al. [6]. Furthermore, accuracy in [6] with 512 partitions was achieved by the *4SBM* method featuring Boole's quadrature with just 32 partitions. In addition, Boole's quadrature performed better than Simpson's quadrature, as expected. Consistency is again displayed among all candidates of CNC quadrature considered in solving Problem 2, as presented in Table 3, confirming the accuracy of the *4SBM* solution technique as an efficient LFIDE solver. Comparatively, we checked for the efficiency of our derived *4SBM* with a semi-analytical solution approach of Mohd et al. [7] for h = 0.0625, taking into account the Maximum Absolute Error (Max. AE) recorded in the process. Findings are reported in Table 4. Performance results displayed in Table 4 indicated improvement over the technique described in [7]. In addition, our method addresses a known setback usually suffered by methods such as that of [7], in which convergence is easily achieved if the value of the independent parameter is very small. Varying n in the solution process of Problem 3, we attempted to determine a sufficient number of partitions necessary to achieve a desirable level of convergence. This optimization test reported in Table 5 revealed that it is not necessarily required to excessively partition certain problems to achieve a significant level of

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convergence. Thus, our solution methods presented in this work are considered optimal. In this case, it is observed that a lesser cost of function evaluations (with $n = 12$) resulted in a milder deviation from a theoretical solution of Problem 3 when the Max. AE encountered in each solution instance of n is considered. Finally, *4SBM* with ONC for $n = 3$ was implemented. Also, we examined the performances of the ONC quadrature for $n = 2$ (Milne's Rule) and CNC quadrature for $n = 3$ (Simpson's 3/8 Rule) using the derived *3SBM*. Findings are presented in Tables 6, 7 and 8 respectively. The numerical simulation reported in Table 6 is performed at 15-digit precision. This result justifies the possibility of implementing various classes of Newton-Côtes quadrature, with appropriate step length h, while the efficiency of *4SBM* is re-established. The result in Table 7 demonstrates the possibility of adopting any quadrature candidate once an appropriate step method is employed. Milne's Rule performed as well as Simpson's 3/8 Rule, as seen in Tables 7 and 8. In all cases, the deviation from the exact solution in Tables 6, 7, and 8 confirms the accuracy of *3SBM*, just like the *4SBM*, despite a few numbers of partitions.

5. Conclusion

In summary, the study reviewed the derivation and implementation of an efficient multistep method and a trial method as Linear Second-Order Fredholm Integro-Differential Equation (LFIDE) solvers. These solvers were applied with integration rules from closed and open Newton-Cotes quadrature, demonstrating promising and convergent results in solving test problems. The significance of calculating the appropriate step size h, for any chosen quadrature and the correlation between the step number of the multistep method and the number of partitions were emphasized. Our finding also revealed that increasing digit precision minimizes errors, and selecting large values for m may not be necessary, as indicated in Table 5 for Problem 3, where a tolerable error is achieved with an appropriate h and efficient quadrature. The research suggests potential directions for extension:

1. Exploring Volterra or Fredholm-Volterra type linear and non-linear higher-order equations.
2. Investigating the replacement of Newton-Cotes quadrature with other families of integration quadrature.
3. Exploring the application of quantum algorithms for solving differential equations on near-term quantum machines.

6. Conflict of Interest

The authors have not declared any conflict of interest.

7. Acknowledgment

We wish to state that all cited materials are duly referenced.

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